The Kac-Ward approach to the Ising Model

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Multi-point correlations and dual boundary conditions

3.1 Introduction and main results

As we have seen in Chapter 2, the partition function of the Ising model on a planar graph $\mathcal{G}$ with positive boundary conditions is proportional to the even subgraph generating function $Z_{\mathcal{G}^*}(x)$, where $\mathcal{G}^*$ is the weak dual of $\mathcal{G}$ and x is the weight vector defined by the low-temperature expansion. Similarly, the partition function of the model with free boundary conditions is proportional to $Z_{\mathcal{G}}(x)$, where x is the high-temperature weight vector. This is the famous Kramers–Wannier duality of the planar Ising model. We used it together with Theorem 2.10 to express the free energy density and the correlation functions in terms of signed loops. Then, by analyzing the asymptotic growth rate of the signed loops, we were able to show analyticity of the free energy density and describe the behaviour of the correlation functions for off-critical temperatures. However, the low-temperature loop expansions work only for $\beta \in (\beta_c, \infty)$ and the high-temperature loop expansions are valid only for $\beta \in (0, \beta_c)$. This is because for $\beta = \beta_c$, the spectral radius of the associated transition matrices (2.13) becomes 1 in the thermodynamic limit, and the loop expansions work only when this spectral radius is smaller than 1. Hence, our formulas for the Ising model with positive boundary conditions concerned only the low-temperature regime and the results for the model with free boundary conditions were true only at high temperatures.

This was not an issue for the free energy density since, by a standard argument, it is independent of boundary conditions. The main aim of this chapter is to prove the analogous independence for the two-point functions. To this end, we will analyze expansions for the Ising model with the dual boundary conditions, i.e. we will use the high-temperature expansion for positive boundary conditions and the low-temperature expansion for free boundary conditions. This will result in more complicated formulas for the correlation functions. They will be expressed in terms of sums of generating functions of graphs with odd degrees at certain vertices and even degrees everywhere else. To study such generating functions, in Section 3.2, we introduce the notions of pinned generating functions of even subgraphs and loops, and we describe their properties. The pinned generating functions can be seen as generalizations of the generating functions used in the proof of Theorem 2.7. In Sections 3.4, 3.5 and 3.6, we show how the correlation functions can be expressed in terms of pinned generating functions. All these considerations will lead to a result dual to Theorem 2.7 (see Section 2.1.2 for the notation):

Theorem 3.1. *For all $\beta \in (0, \beta_c)$ and $u, v \in \mathbb{Z}^2$ ($u \neq v$),*

$$\lim_{\mathcal{G} \to \mathbb{Z}^2} \langle \sigma_u \sigma_v \rangle^+_{\mathcal{G},\beta} = \left(\sum_{r=1}^{\infty} f_r^{uu^*}(x'_\gamma) \right) \langle \sigma_{u^*} \sigma_{v^*} \rangle^{\text{free}}_{\mathbb{Z}^{2*},\beta^*} =: \langle \sigma_u \sigma_v \rangle^+_{\mathbb{Z}^2,\beta}.$$

As a corollary to Theorems 2.7 and 3.1, we will obtain that the off-critical two-point functions are independent of boundary conditions:

Corollary 3.2. *For all $\beta \neq \beta_c$ and $u, v \in \mathbb{Z}^2$,*

$$\langle \sigma_u \sigma_v \rangle^+_{\mathbb{Z}^2,\beta} = \langle \sigma_u \sigma_v \rangle^{\text{free}}_{\mathbb{Z}^2,\beta}.$$

Note that this is a classical result and we only provide an alternative derivation of it. A proof of the above equality for all values of β can be found e.g. in [43], where it is given for general Ising models with periodic interactions. It uses the fact that the free energy density limit is continuously differentiable in β. Alternative proofs which use the random cluster representation of the Ising model can be found in [24, 30]. In our approach, we are able to prove this identity only for non-critical temperatures. This is because the loop expansions at criticality reach their radius of convergence, and it is difficult to analyze infinite series at their radius of convergence. Nonetheless, our methods can be used to prove analogous results for aperiodic Ising models (see Chapter 4) and the combinatorial identities presented in this chapter yield other interesting properties of the correlation functions (see e.g. Theorem 3.3).

For simplicity of the exposition, we have so far considered Ising models on rectangular subgraphs of the square lattice. However, the combinatorial notions developed in Chapter 2 work for arbitrary graphs in the plane. In this chapter, we will present several side results about the correlation functions of the Ising model on a general graph $\mathcal{G}$ embedded in the plane without edge crossings. Hence, we will need a definition of the Ising measure (2.1) which is suitable for this setting.

To this end, let $\partial\mathcal{G}$ be the set of vertices incident to the unbounded face of $\mathcal{G} = (V, E)$, and let $\Omega^+_{\mathcal{G}}$ and $\Omega^{\text{free}}_{\mathcal{G}}$ be as in (2.1). Until now, we have assumed that the interaction between spins is uniform and given by the inverse temperature β. To model non-homogenous interactions between spins, we introduce a vector of positive (*ferromagnetic*) *coupling constants* $(J_e)_{e \in E}$. The Ising model with coupling constants J is given by the following probability measure on the space of spin configurations:

$$\mathbf{P}^{\square}_{\mathcal{G},\beta}(\sigma) = \frac{1}{\mathcal{Z}^{\square}_{\mathcal{G},\beta}} \prod_{uv \in E} e^{\beta J_{uv} \sigma_u \sigma_v}, \qquad \sigma \in \Omega^{\square}_{G}, \tag{3.1}$$

where $\square \in \{\text{free}, +\}$ as before, and $Z^{\square}_{\mathcal{G},\beta}$ is the partition function, i.e.

$$\mathcal{Z}^{\square}_{\mathcal{G},\beta} = \sum_{\sigma\in\Omega^{\square}} \prod_{uv\in E} e^{\beta J_{uv}\sigma_u\sigma_v}. \tag{3.2}$$

Note that the homogenous Ising model from the previous chapter is recovered by taking $J \equiv 1$.

Let $A \subset V$ and $\sigma_A = \prod_{v\in A}\sigma_v$. Recall that by $\langle\sigma_A\rangle^{\square}_{\mathcal{G},\beta}$, we denote the average of σ_A with respect to $\mathbf{P}^{\square}_{\mathcal{G},\beta}$. If A contains more than two spins, we call this average a *multi-point* function. As a side corollary of the properties of pinned generating functions, we will rederive in Section 3.4 the following, interesting in itself, result about boundary spin correlation functions:

Theorem 3.3 (Boel & Groeneveld & Kasteleyn, [7]). *Let A be a set of vertices lying on the boundary of one face of $\mathcal{G}$, such that $|A|$ is even. Then,*

$$\langle\sigma_A\rangle^{\text{free}}_{\mathcal{G},\beta} = \sum_{\pi\in\mathcal{P}(A)} \operatorname{sgn}(\pi) \prod_{\{u,v\}\in\pi} \langle\sigma_u\sigma_v\rangle^{\text{free}}_{\mathcal{G},\beta}.$$

Above, the sum is taken over all pairings of the vertices in A, and $\operatorname{sgn}\pi = \pm 1$, depending on the parity of the number of crossings induced by the paring π. For precise definitions, see the next section.

This chapter is organised as follows: in Section 3.2, we introduce the notions of pinned generating functions and state their main properties. In Sections 3.3 and 3.4, we shortly discuss the multi-point correlation functions, and using the results from Section 3.2, we provide a new proof of Theorem 3.3. In Sections 3.5 and 3.6, we present formulas for the correlation functions with the dual boundary conditions. In Section 3.7 we prove the combinatorial results and in Section 3.8, we prove the results for the Ising model.

3.2 Stars and pinned generating functions

Let $\mathcal{G} = (V, E)$ be a graph in the plane. We assume that all its edges are representative in the sense of Section 2.1.3. Let $v_\bullet$ be a fixed additional vertex, i.e. $v_\bullet \notin V$, and let $A \subset V$ be a set of even cardinality. By $\mathcal{S}(v_\bullet, A)$ or simply by $\mathcal{S}(A)$, we denote the *star graph* $\big(\{v_\bullet\}\cup A, \bigcup_{v\in A}\{v_\bullet v\}\big)$. We assume that the edges of $\mathcal{S}$ are additional in the sense of Section 2.1.3. Moreover, we assume that the edges of $\mathcal{S}$ may be piecewise linear (we do not want to assign any particular vertex to the turning points of an edge) and that no two edges cross each other. For instance, Figures 3.1 and 3.2 below show star graphs with piecewise linear edges. However, when we will talk about transition matrices for graphs containing a star $\mathcal{S}$, we will mean the standard transition matrix (2.13) of the graph with vertices added at all turning points of the edges of $\mathcal{S}$.

Let $E_{\mathcal{S}}$ be the edge set of $\mathcal{S}$ and let $x = (x_e)_{e\in E\cup E_{\mathcal{S}}}$ be a complex weight vector. We denote by $Z_{\mathcal{G},\mathcal{S}}(x)$ the *generating function of even subgraphs* of $\mathcal{G}\cup\mathcal{S}$ *pinned at* $\mathcal{S}$, i.e.

$$Z_{\mathcal{G},\mathcal{S}}(x) = \sum_{\substack{E_{\mathcal{S}}\subset F\subset E\cup E_{\mathcal{S}} \\ F \text{ even}}} (-1)^{C(F)} \prod_{e\in F} x_e, \tag{3.3}$$

where $C(F)$ is as in (2.10). Note that the condition that $|A|$ is even is necessary for the above sum to be non-trivial.

We say that an unordered configuration of loops $\{\ell_1, \dots, \ell_s\}$ is *pinned at* $\mathcal{S}$ and we write $\{\ell_1, \dots, \ell_s\} \vdash \mathcal{S}$ if each loop ℓ_i goes through at least one edge from $\mathcal{S}$, and each edge in $\mathcal{S}$ is traversed exactly once by exactly one loop ℓ_i. Note that if $\{\ell_1, \dots, \ell_s\}$ is pinned at $\mathcal{S}$, then all the loops ℓ_i are distinct and have multiplicity 1. We define the *generating function of loops* in $\mathcal{G} \cup \mathcal{S}$ *pinned at* $\mathcal{S}$ by the formal sum

$$\Phi_{\mathcal{G},\mathcal{S}}(x) = \sum_{r=1}^{\infty} \sum_{s=1}^{|E_{\mathcal{S}}|/2} \sum_{\substack{\{\ell_1,\dots,\ell_s\}\vdash \mathcal{S} \\ r(\ell_1)+\dots+r(\ell_s)=r}} \prod_{i=1}^{s} w(\ell_i; x),$$

where the loops ℓ_i are in $\mathcal{G} \cup \mathcal{S}$. Recall from see Section 2.1.3 that $r(\ell)$ is the number of representative edges (with multiplicities) which are traversed by ℓ.

Let $\rho_{\mathcal{G}}(x)$ be the spectral radius of the transition matrix $\Lambda_{\mathcal{G}}(x)$. The following results present the basic properties of the generating functions defined above.

Lemma 3.4. *Let $\mathcal{S}$ be as above. If $\rho_{\mathcal{G}}(x) < 1$, then*

$$\Phi_{\mathcal{G},\mathcal{S}}(x) = \frac{Z_{\mathcal{G},\mathcal{S}}(x)}{Z_{\mathcal{G}}(x)}.$$

Note that the above result implies that $\Phi_{\mathcal{G},\mathcal{S}}(x)$ is convergent whenever $\rho_{\mathcal{G}}(x) < 1$.

Recall from Section 2.2.1 that a pairing at $v_\bullet$ is a partition of A into sets of size 2. We denote by $\mathcal{P}(A)$ the collection of all pairings at $v_\bullet$. If $\pi \in \mathcal{P}(A)$, then we put $\operatorname{sgn}(\pi) = -1$ if the number of crossings in $\mathcal{S}$ induced by π at the vertex $v_\bullet$ is odd, and $\operatorname{sgn}(\pi) = 1$ otherwise (see the proof of Proposition 2.12). Note that, since the edges may be piecewise linear, the sign of a pairing depends on the way $\mathcal{S}$ is embedded in the plane. If $\mathcal{S}$ has only two arms, i.e. $A = \{u, v\}$, then to avoid unnecessary brackets, we will write $\mathcal{S} = \mathcal{S}(u, v)$.

Lemma 3.5. *Let $\mathcal{S}$ be as above. If $\rho_{\mathcal{G}}(x) < 1$, then*

$$\Phi_{\mathcal{G},\mathcal{S}}(x) = \sum_{\pi\in\mathcal{P}(A)} \operatorname{sgn}(\pi) \prod_{\{u,v\}\in\pi} \Phi_{\mathcal{G},\mathcal{S}(u,v)}(x). \tag{3.4}$$

Let $\mathcal{S} = \mathcal{S}(A)$ and $\mathcal{S}' = \mathcal{S}(A')$ be two stars such that A and A' are disjoint, and such that the edges of S and S' do not cross each other.

Lemma 3.6. *Let $\mathcal{S}$ and $\mathcal{S}'$ be as above. Let $\pi \in \mathcal{P}(A)$ and $\pi' \in \mathcal{P}(A')$. Then, the value of*

$$\operatorname{sgn}(\pi \cup \pi') \operatorname{sgn}(\pi) \operatorname{sgn}(\pi'),$$

where $\operatorname{sgn}(\pi \cup \pi')$ *is computed in the star* $\mathcal{S} \cup \mathcal{S}'$*, is independent of the choice of* π *and* π'*.*

Define $\operatorname{sgn}(\mathcal{S}, \mathcal{S}') = \operatorname{sgn}(\pi \cup \pi') \operatorname{sgn}(\pi) \operatorname{sgn}(\pi')$, where $\pi \in \mathcal{P}(A)$ and $\pi' \in \mathcal{P}(A')$. This is well defined by the lemma above. Let $\mathcal{P}(A, A') = \{\pi \cup \pi' : \pi \in \mathcal{P}(A),\ \pi' \in \mathcal{P}(A')\} \subset \mathcal{P}(A \cup A')$. By splitting the sum in Lemma 3.5 into a sum over the pairings from $\mathcal{P}(A, A')$ and the rest and using Lemma 3.6, we obtain the following corollary:

Corollary 3.7. *Let* $\mathcal{S}$ *and* $\mathcal{S}'$ *be as above. If* $\rho_{\mathcal{G}}(x) < 1$*, then*

$$\Phi_{\mathcal{G},\mathcal{S}\cup\mathcal{S}'}(x) = \operatorname{sgn}(\mathcal{S},\mathcal{S}')\Phi_{\mathcal{G},\mathcal{S}}(x)\Phi_{\mathcal{G},\mathcal{S}'}(x) + \sum_{\substack{\pi\in\mathcal{P}(A\cup A')\\ \pi\notin\mathcal{P}(A,A')}} \operatorname{sgn}(\pi) \prod_{\{v,u\}\in\pi} \Phi_{\mathcal{G},\mathcal{S}(v,u)}(x).$$

Remark 3.8. *We said that the pinned generating functions are generalizations of the functions used in the proof of Theorem 2.7. Indeed, note the similarity of* (3.3) *to the right-hand side of* (2.46)*, where the path* γ *from Theorem 2.7 is seen as a star* $\mathcal{S}$ *with two arms. Then, the generating function* $\Phi_{\mathcal{G},\mathcal{S}}(x) = \Phi_{\mathcal{G},\gamma}(x)$ *is the sum of signed weights of single loops which go through* γ *exactly once. This sum appears in the statements of Theorems 2.7 and 3.1.*

3.3 Low-temperature multi-point functions

The generalization of the low-temperature formulas with positive boundary conditions (2.40) and (2.41) to multi-point functions is straight-forward. To make this exposition more complete, we will briefly present it in this section. To this end, fix a vertex $v_\bullet \notin V$ and let $A \subset V \setminus \partial\mathcal{G}$. If $|A|$ is even, then choose a star graph $\mathcal{S} = \mathcal{S}(A)$. Otherwise, choose a vertex $v \in \partial\mathcal{G}$ and a graph $\mathcal{S} = \mathcal{S}(\{v\} \cup A)$. Recall that by $\mathcal{G}^* = (V^*, E^*)$, we denote the weak dual graph of $\mathcal{G}$ and by e^*, the only edge of $\mathcal{G}^*$ crossing e. To each even subgraph F of $\mathcal{G}^*$, there bijectively corresponds a spin configuration $\sigma(F) \in \Omega^+_{\mathcal{G}}$, such that $(uv)^* \in F$ if and only if $\sigma_u(F) \neq \sigma_v(F)$. Note that $\sigma_A(F) = -1$ if the edges of $\mathcal{S}$ cross F an odd number of times, and $\sigma_A(F) = 1$ otherwise. This yields

$$\langle \sigma_A \rangle^+_{\mathcal{G},\beta} = \Bigg(\sum_{\substack{F\subset E^*\\ F \text{ even}}} \sigma_A(F) \prod_{e\in F} x_e \Bigg) \Bigg/ \Bigg(\sum_{\substack{F\subset E^*\\ F \text{ even}}} \prod_{e\in F} x_e \Bigg) = \frac{Z_{\mathcal{G}^*}(x')}{Z_{\mathcal{G}^*}(x)}, \tag{3.5}$$

where $x_{e^*} = \exp(-2\beta J_e)$, and $x'_{e^*} = -x_{e^*}$ if e^* is crossed an odd number of times by the edges of $\mathcal{S}$ and $x_{e^*} = x'_{e^*}$ otherwise.

We say that a loop in $\mathcal{G}^*$ is A-odd if its total winding number around the vertices of A is odd. Equivalently, a loop is A-odd if it crosses the edges of $\mathcal{S}$ an odd number of times, where $\mathcal{S}$ is any star constructed as above. From Theorem 2.10, (3.5) and the definition of the weight vector x', it follows that if $\rho_{\mathcal{G}^*}(x) < 1$ and $\rho_{\mathcal{G}^*}(x') < 1$, then

$$\langle \sigma_A \rangle^+_{\mathcal{G},\beta} = \exp\Bigg(-2 \sum_{r=1}^{\infty} \sum_{\ell\in\mathcal{L}^A_r(\mathcal{G}^*)} w(\ell; x) \Bigg), \tag{3.6}$$

where $\mathcal{L}_r^A(\mathcal{G}^*)$ is the set of all A-odd loops of length r in $\mathcal{G}^*$.

Remark 3.9. *Using* (3.6) *and the arguments from the proof of Corollary 2.6, one can prove that in the low-temperature Ising model on the square lattice, the spins decorrelate exponentially fast, i.e. for $\beta \in (\beta_c, \infty)$, $\langle \sigma_A \sigma_{A'} \rangle^+_{\mathbb{Z}^2,\beta}$ is close to $\langle \sigma_A \rangle^+_{\mathbb{Z}^2,\beta} \langle \sigma_{A'} \rangle^+_{\mathbb{Z}^2,\beta}$ with the error being exponentially small with respect to the distance between the sets of vertices A and A'. The idea of the proof is as before: if A and A' are far away from each other, then the loops which are simultaneously A-odd and A'-odd have to be long since they have to wind around the vertices from both A and A'. Using the bounds on the spectral radius from Theorem 2.11, one can prove that the total signed weight of such loops is exponentially small.*

3.4 High-temperature multi-point functions

In this section, we assume that $|A|$ is even. Otherwise, the considered correlation functions are trivially zero. Performing the high-temperature expansion for $\langle \sigma_A \rangle^{\text{free}}_{\mathcal{G},\beta}$ as explained in Section 2.4.1, we arrive at a formula involving a generating function of subgraphs of $\mathcal{G}$, which have odd degrees at the vertices from A and even degrees everywhere else, i.e.

$$\langle \sigma_A \rangle^{\text{free}}_{\mathcal{G},\beta} = \frac{1}{Z_{\mathcal{G}}(x)} \sum_{\substack{F \subset E \\ \delta F = A}} \prod_{e \in F} x_e, \tag{3.7}$$

where δF is the set of vertices with odd degree in the graph (V, F), and $x_e = \tanh \beta J_e$. We can translate this formula into the language of pinned generating functions. To this end, fix $v_\bullet \notin V$ and a star graph $\mathcal{S} = \mathcal{S}(A)$. Then,

$$\langle \sigma_A \rangle^{\text{free}}_{\mathcal{G},\beta} = \frac{Z_{\mathcal{G},\mathcal{S}}(x')}{Z_{\mathcal{G}}(x)}, \tag{3.8}$$

where x is as above, $x'_{v_\bullet v} = 1$ for any $v \in A$, $x'_e = -x_e$ if $e \in E$ and e is crossed an odd number of times by the edges of $\mathcal{S}$, and $x_e = x'_e$ otherwise. As before, the sign changes in the weight vector are to compensate for the signs in the definition of the pinned generating function. Using Lemma 3.4, we obtain that if $\rho_{\mathcal{G}}(x') < 1$, then

$$\langle \sigma_A \rangle^{\text{free}}_{\mathcal{G},\beta} = \Phi_{\mathcal{G},\mathcal{S}}(x') \frac{Z_{\mathcal{G}}(x')}{Z_{\mathcal{G}}(x)}. \tag{3.9}$$

Note that by (3.5), as in the case of the two-point function (2.49), the above ratio of graph generating functions, can be interpreted as a multi-point function for positive boundary conditions on a graph, whose weak dual is $\mathcal{G}$.

Remark 3.10. *Using* (3.9) *and Corollary 3.7, one can prove that in the high-temperature Ising model on the square lattice, the spins also decorrelate exponentially fast, i.e. for*

$\beta \in (0, \beta_c)$, $\langle \sigma_A \sigma_{A'} \rangle^{\text{free}}_{\mathbb{Z}^2,\beta}$ is close to $\langle \sigma_A \rangle^{\text{free}}_{\mathbb{Z}^2,\beta} \langle \sigma_{A'} \rangle^{\text{free}}_{\mathbb{Z}^2,\beta}$ with the error being exponentially small with respect to the distance between the sets of vertices A and A'. The idea of the proof is as follows: by (3.9) *and Remark 3.9, it is enough to prove an analogous statement for the functions $\Phi_{\mathcal{G},\mathcal{S}(A\cup A')}$ and $\Phi_{\mathcal{G},\mathcal{S}(A)}\Phi_{\mathcal{G},\mathcal{S}(A')}$ (with appropriate weight vectors). By Corollary 3.7, the difference between these two is expressed in terms of two-point pinned generating functions involving loops of length at least the distance between A and A'. Using the bound on the operator norm of the transition matrix from Theorem 2.11, we conclude that that this difference is exponentially small. Note that a similar reasoning will be used in the proof of Theorem 3.1.*

Suppose now that all vertices from A lie on the boundary of one face of $\mathcal{G}$. Let $v_\bullet$ lie inside this face and let $\mathcal{S}$ be such that its edges do not cross any edge from $\mathcal{G}$. In this case, for $\pi \in \mathcal{P}(A)$, $\operatorname{sgn}(\pi)$ is independent of the embedding of $\mathcal{S}$. By (3.9), we have that if $\rho_{\mathcal{G}}(x) < 1$, then

$$\langle \sigma_A \rangle^{\text{free}}_{\mathcal{G},\beta} = \Phi_{\mathcal{G},\mathcal{S}}(x). \tag{3.10}$$

The proof of Theorem 3.3 is now very short:

Proof of Theorem 3.3. By Lemma 3.5 and (3.10), if $\rho_{\mathcal{G}}(x) < 1$, where $x_e = \tanh \beta J_e$, then the desired identity holds true. Since the spectral radius is continuous in β, it is smaller than 1 on some open neighbourhood of 0. Since on both sides we have real analytic functions of β which agree on the neighborhood of 0, by uniqueness of the analytic continuation, they agree on the whole interval $(0, \infty)$. □

Note that Theorem 3.3 was also proved in [32, 33] in the setting of correlation functions and fermionic observables on the square lattice. Since these observables are given by generating functions which are similar to Φ (see Chapter 5), our methods allow to generalize the result of [32, 33] to all planar graphs.

3.5 Low-temperature correlations with free boundary conditions

In the low-temperature expansion, one sums over the subgraphs of the dual graph which define boundaries between clusters of positive and negative spins in a spin configuration on the primal graph. In the case of positive boundary condition, this resulted in summing over all even subgraphs of the *weak* dual graph. With free boundary conditions, the contours that arise in this method are the even subgraphs of the *full* dual graph. This is problematic since usually, the full dual graph has one vertex of a very high degree, which in turn can result in a large spectral radius of the transition matrix for this graph. We want to use loop expansions and these work under the condition of the spectral radius being smaller than 1. Hence, we will have to work with the pinned generating functions rather than the even subgraph generating function for the full dual graph. The pinned generating functions have loop expansions under an intuitively weaker condition, namely that the spectral radius of the transition matrix for the weak dual graph is smaller than 1.

Let $A \subset V$ be of even cardinality and let $\mathcal{G}^\star = (V^\star, E^\star)$ be such that its weak dual is $\mathcal{G}$ (see Figure 3.1). To each $F \subset E^\star$ satisfying $\delta F \subset \partial\mathcal{G}^\star$ and $bb' \notin F$ for all $b, b' \in \partial\mathcal{G}^\star$, there correspond exactly two, equally probable, spin configurations $\sigma(F), \sigma'(F) \in \Omega_{\mathcal{G}}^{\text{free}}$ satisfying $\sigma(F) = -\sigma'(F)$, for which F defines the boundaries between clusters of positive and negative spins. Since $|A|$ is even, $\sigma_A(F) = \sigma'_A(F)$. Hence, without changing the value of the correlation function, we can choose an arbitrary vertex and condition on the spin there being positive. After conditioning, to each F there corresponds exactly one of the spin configurations $\sigma(F), \sigma'(F)$. From the low-temperature expansion, we have

$$\langle \sigma_A \rangle_{\mathcal{G},\beta}^{\text{free}} = \Bigg(\sum_{B \subset \partial\mathcal{G}^\star} \sum_{\substack{F \subset E^\star \\ \delta F = B}} \sigma_A(F) \prod_{e \in F} x_e \Bigg) \Big/ \Bigg(\sum_{B \subset \partial\mathcal{G}^\star} \sum_{\substack{F \subset E^\star \\ \delta F = B}} \prod_{e \in F} x_e \Bigg),$$

where $x_e = \exp(-2J_{e^*}\beta)$ if $e^* \in E$, and $x_{bb'} = 0$ if $b, b' \in \partial\mathcal{G}^\star$.

We want to express this formula in terms of pinned generating functions. To this end, fix a vertex $v_\bullet$ somewhere in the unbounded face of $\mathcal{G}^\star$, and let $\mathcal{S} = \mathcal{S}(v_\bullet, \partial\mathcal{G}^\star)$ be a star whose edges do not cross any edge from $E^\star$. Moreover, fix a star graph $\mathcal{S}' = \mathcal{S}(v_*, A)$ for some vertex v_* (it can be $v_\bullet$ or some other additional vertex). If $B \subset \partial\mathcal{G}^\star$, then we will assume that the embedding of $\mathcal{S}(B) = \mathcal{S}(v_\bullet, B)$ uses the appropriate subset of the edges drawn for S. If F is as above, then $\sigma_A(F) = -1$ if the edges of $\mathcal{S}'$ cross the edges of F an odd number of times, and $\sigma_A(F) = 1$ otherwise. Hence,

$$\langle \sigma_A \rangle_{\mathcal{G},\beta}^{\text{free}} = \Big(\sum_{B \subset \partial\mathcal{G}^\star} Z_{\mathcal{G}^\star, \mathcal{S}(B)}(x') \Big) \Big/ \Big(\sum_{B \subset \partial\mathcal{G}^\star} Z_{\mathcal{G}^\star, \mathcal{S}(B)}(x) \Big), \tag{3.11}$$

where $x_{vv_\bullet} = 1$ for any v, and x restricted to $E^\star$ is as above. The modified weight vector x' satisfies: $x'_e = -x_e$ if e is crossed an odd number of times by the edges of S', and $x'_e = x_e$ otherwise.

3.6 High-temperature correlations with positive boundary conditions

Let $A \subset V \setminus \partial\mathcal{G}$. For positive boundary conditions, the subgraphs of $\mathcal{G}$ that appear in the high-temperature expansion of the correlation function $\langle \sigma_A \rangle_{\mathcal{G},\beta}^{+}$ have, as before, odd degrees at all vertices from A. However, unlike in the case of free boundary conditions, they can also have odd degrees at the vertices from $\partial\mathcal{G}$. This is due to the fact that the boundary spins are fixed. Hence, graphs which have odd degrees on the boundary do not vanish from the expansion when one interchanges the order of summation as it

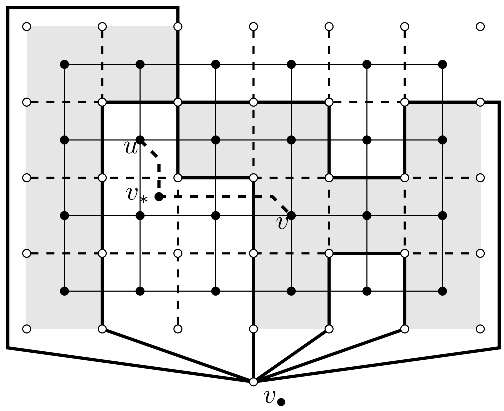

Figure 3.1: A rectangle $\mathcal{G}$ with its graph $\mathcal{G}^\star$ (dashed edges). The bold edges show a graph $\mathcal{S}(B)$ appearing in the low-temperature expansion for free boundary conditions. The removed edges of $\mathcal{G}^\star$ correspond to the zeroed coordinates of the weight vectors x and x'. The shaded area indicates positive spins. The star graph $S(v_*, \{u, v\})$ drawn in bold dashed lines is used to define the weight vector x'.

was done in (2.32). Also note that the edges connecting two boundary vertices of $\mathcal{G}$ are immaterial since they contribute a common constant factor to the probability of all spin configurations. Hence, these edges get a zero weight in the expansion:

$$\langle \sigma_A \rangle^+_{\mathcal{G},\beta} = \Big(\sum_{B \subset \partial \mathcal{G}} \sum_{\substack{F \subset E \\ \delta F = A \cup B}} \prod_{e \in F} x_e \Big) \Big/ \Big(\sum_{B \subset \partial \mathcal{G}} \sum_{\substack{F \subset E \\ \delta F = B}} \prod_{e \in F} x_e \Big),$$

where $x_{bb'} = 0$ if $b, b' \in \partial \mathcal{G}$, and $x_e = \tanh \beta J_e$ otherwise.

We will translate this formula to the language of pinned generating functions. To this end, fix a vertex $v_\bullet$ somewhere in the unbounded face of $\mathcal{G}$ and let $\mathcal{S} = \mathcal{S}(A)$. If $B \subset \partial \mathcal{G}$, then we assume that the graph $\mathcal{S}(B)$ does not have any edge crossings with $\mathcal{G}$ and $\mathcal{S}$. Using the above formula, we get

$$\langle \sigma_A \rangle^+_{\mathcal{G},\beta} = \Big(\sum_{B \subset \partial \mathcal{G}} Z_{\mathcal{G}, \mathcal{S} \cup \mathcal{S}(B)}(x') \Big) \Big/ \Big(\sum_{B \subset \partial \mathcal{G}} Z_{\mathcal{G}, \mathcal{S}(B)}(x) \Big), \tag{3.12}$$

where $x_{v_\bullet v} = 1$ for all v, and x restricted to E is as above. The modified weight vector x' satisfies: $x'_e = -x_e$ if e is crossed by the edges from $\mathcal{S}(A)$ an odd number of times, and $x'_e = x_e$ otherwise.

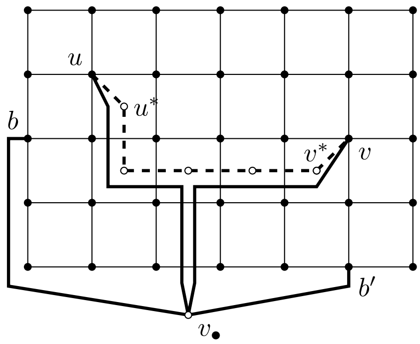

Figure 3.2: A path γ connecting u and v as in Figure 2.1, and a star graph $\mathcal{S} = \mathcal{S}(\{u, v, b, b'\})$ used in the high-temperature expansion for positive boundary conditions. The weight vector x' defined for $\mathcal{S}$ in (3.12) and the weight vector x'_γ defined in (2.45) are in this case equal on the edges of the rectangle.

3.7 Proofs of the combinatorial identities

Proof of Lemma 3.6. Note that $\operatorname{sgn}(\pi \cup \pi')\operatorname{sgn}(\pi)\operatorname{sgn}(\pi')$ is negative if and only if the number of crossings at $v_\bullet$ between the walks induced by π and π' is odd. Also note that if u, v, w, z, u', v' are distinct, then the number of crossings between the walks $(u, v_\bullet, v)$, $(w, v_\bullet, z)$ and the walk $(u', v_\bullet, v')$ has the same parity as the number of crossings between the walks $(u, v_\bullet, z)$, $(w, v_\bullet, v)$ and the walk $(u', v_\bullet, v')$. Starting with a fixed pairing, one can obtain any other pairing by repeatedly interchanging connections as above. Hence, the claim of the lemma follows. □

Proof of Lemma 3.4. We will use Theorem 2.10 and the following formula:

$$Z_{\mathcal{G},\mathcal{S}}(x) = \prod_{e \in E_{\mathcal{S}}} x_e \Big[\Big(\prod_{e \in E_{\mathcal{S}}} \frac{\partial}{\partial x_e}\Big) Z_{\mathcal{G}\cup\mathcal{S}}(x)\Big]\Big|_{x_e=0,\ e\in E_{\mathcal{S}}}. \tag{3.13}$$

Let ρ_0 be such that $\rho = \rho_{\mathcal{G}}(x) < \rho_0 < 1$. By continuity of the spectral radius, we can choose t so small that $\rho_{\mathcal{G}\cup\mathcal{S}}(y) < \rho_0$ whenever $y \in B_t$, where

$$B_t = \{y \in \mathbb{C}^{E\cup E_{\mathcal{S}}} : y_e = x_e \text{ for } e \in E, \text{ and } |y_e| < t \text{ for } e \in E_{\mathcal{S}}\}.$$

Suppose that $y \in B_t$ and let

$$f_r(y) = \sum_{\ell \in \mathcal{L}_r(\mathcal{G})} w(\ell; y) \text{ and } g_r(y) = \sum_{\ell \in \mathcal{L}^c_r(\mathcal{G})} w(\ell; y),$$

where $\mathcal{L}_r(\mathcal{G})$ is the set of all loops of length r in $\mathcal{G}$ and $\mathcal{L}^c_r(\mathcal{G}) = \mathcal{L}_r(\mathcal{G}\cup\mathcal{S}) \setminus \mathcal{L}_r(\mathcal{G})$. As in (2.21), we can express $f_r(y)$ and $g_r(y)$ in terms of the eigenvalues of $\Lambda_{\mathcal{G}}(y)$ and

$\Lambda_{\mathcal{G}\cup\mathcal{S}}(y)$, and we conclude that $|f_r(y)| \leq 2|E|\rho^r$ and $|g_r(y)| \leq 4|E|\rho_0^r$. Similarly to the proof of Theorem 2.9, it follows that

$$|h_{r,s}(y)| := \frac{1}{s!}\Bigg| \sum_{\substack{(\ell_1,\dots,\ell_s)\in(\mathcal{L}^c(\mathcal{G}))^s \\ r(\ell_1)+\dots+r(\ell_s)=r}} \prod_{i=1}^{s} w(\ell_i;y) \Bigg| \leq \frac{4^s|E|^s}{s!}\binom{r-1}{s-1}\rho_0^r$$

uniformly for all $y \in B_t$. This, together with Theorem 2.10, allows us to write

$$\begin{aligned} Z_{\mathcal{G}\cup\mathcal{S}}(y) &= \exp\Bigg(\sum_{r=1}^{\infty} f_r(y) + \sum_{r=1}^{\infty} g_r(y)\Bigg) = \exp\Bigg(\sum_{r=1}^{\infty} g_r(y)\Bigg) Z_{\mathcal{G}}(y) \\ &= \Bigg(1 + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} h_{r,s}(y)\Bigg) Z_{\mathcal{G}}(x). \end{aligned} \tag{3.14}$$

Since $E_{\mathcal{S}}$ does not contain a cycle, $h_{r,s}(y)$ are polynomials and hence holomorphic functions of each of the complex variables x_e, $e \in E_{\mathcal{S}}$. It follows from the above bound that the double series in (3.14) is uniformly convergent on B_t to a holomorphic function in the variables x_e, $e \in E_{\mathcal{S}}$. This allows us, after combining (3.13) and (3.14), to perform the partial differentiation term after term. We arrive at the desired formula after realizing that the factor $1/s!$ from the definition of $h_{r,s}(y)$ vanishes because the only configurations which survive the evaluation at 0 are the configurations pinned at $\mathcal{S}$, and in particular, they are composed of distinct loops. □

Proof of Lemma 3.5. Let $\{\ell_1,\dots,\ell_s\}$ be pinned at $\mathcal{S}$. Note that $\{\ell_1,\dots,\ell_s\}$ induces a pairing at $v_\bullet$, where two vertices are paired when they are connected within $\mathcal{G}$ by one of the loops ℓ_i. To be more precise, let $u \in A$ and let ℓ_i be the unique loop which traverses the edge $v_\bullet u$. Then, u is paired with the vertex $v \in A$, which is the only vertex with the property that ℓ_i traverses a walk of the form $(v_\bullet, u, w_1, \dots, w_n, v, v_\bullet)$ or its reversion, where $w_i \neq v_\bullet$ for $i = 1,\dots,n$. We call this pairing $\pi(\ell_1,\dots,\ell_s)$.

Let $\pi(\ell_1,\dots,\ell_s) = \{\{u_1,v_1\},\dots,\{u_k,v_k\}\}$, where $k = |A|/2$, and let ℓ'_i, $i = 1,\dots,k$, be the loop defined by the closed walk $(v_\bullet, u_i, w_1, \dots, w_n, v_i)$ as above. Note that $\{\ell'_i\}$ is pinned at $\mathcal{S}(u_i,v_i)$ and $\{\ell'_1,\dots,\ell'_k\}$ is pinned at $\mathcal{S}$. Consider the map

$$\{\ell_1,\dots,\ell_s\} \mapsto \{\ell'_1,\dots,\ell'_k\}. \tag{3.15}$$

This assignment of the "refined" configuration of loops is not injective since it does not depend on how the loops ℓ_i traverse the edges of $\mathcal{S}$. Moreover, the inverse image under this map of any configuration $\{\ell'_1,\dots,\ell'_k\}$ of k loops pinned at $\mathcal{S}$ consists of exactly $|\mathcal{P}(A)|$ configurations $\{\ell_1,\dots,\ell_s\}$, also pinned at $\mathcal{S}$, which differ only in the way the loops connect at $v_\bullet$. Let us denote this inverse image by $I(\ell'_1,\dots,\ell'_k)$. Note that in particular $\{\ell'_1,\dots,\ell'_k\} \in I(\ell'_1,\dots,\ell'_k)$.

Suppose that the configuration $\{\ell_1,\dots,\ell_s\}$ is edge disjoint (see Section 2.2.1). Let $C_\bullet(\ell_1,\dots,\ell_s)$ be the total number of vertex crossings between the loops at $v_\bullet$

(including self-crossings), and let $C(\ell_1,\dots,\ell_s)$ be the total number of the remaining vertex and edge crossing between the loops (including self-crossings). Note that $\operatorname{sgn}\big(\pi(\ell'_1,\dots,\ell'_k)\big) = (-1)^{C_\bullet(\ell'_1,\dots,\ell'_k)}$ and $C(\ell_1,\dots,\ell_s) = C(\ell'_1,\dots,\ell'_k)$. From the fact that $\sum_{\pi\in\mathcal{P}(A)}\operatorname{sgn}(\pi) = 1$ (see the proof of Proposition 2.12) and from (2.15), it follows that

$$\begin{aligned}\sum_{\substack{\{\ell_1,\dots,\ell_s\}\\ \in I(\ell'_1,\dots,\ell'_k)}}\prod_{i=1}^{s}\operatorname{sgn}(\ell_i) &= \sum_{\substack{\{\ell_1,\dots,\ell_s\}\\ \in I(\ell'_1,\dots,\ell'_k)}}(-1)^{C_\bullet(\ell_1,\dots,\ell_s)+C(\ell_1,\dots,\ell_s)}\\ &= (-1)^{2C_\bullet(\ell'_1,\dots,\ell'_k)+C(\ell'_1,\dots,\ell'_k)}\\ &= \operatorname{sgn}\big(\pi(\ell'_1,\dots,\ell'_k)\big)\prod_{i=1}^{k}\operatorname{sgn}(\ell'_i).\end{aligned}$$

By considering the winding angles instead of crossings, the outermost identity can be also proved without the assumption of the loops being edge disjoint. Since the map (3.15) does not change the absolute value of the weight of a loop configuration, it follows that for all r,

$$\begin{aligned}&\sum_{s=1}^{k}\sum_{\substack{\{\ell_1,\dots,\ell_s\}\vdash\mathcal{S}\\ r(\ell_1)+\dots+r(\ell_s)=r}}\prod_{i=1}^{s} w(\ell_i;x)\\ &\qquad = \sum_{\pi\in\mathcal{P}(A)}\operatorname{sgn}(\pi)\sum_{r_1+\dots+r_k=r}\prod_{i=1}^{k}\Bigg(\sum_{\substack{\{\ell'\}\vdash\mathcal{S}(u_i,v_i)\\ r(\ell')=r_i}} w(\ell';x)\Bigg),\end{aligned}$$

where $\pi = \{\{u_1,v_1\},\dots,\{u_k,v_k\}\}$. This means that the left hand side of (3.4) is a sum of the Cauchy products of the series on the right hand side.

Now, consider a weight vector tx where $t > 0$ is a scaling factor (we scale only the coordinates corresponding to E). If t is small enough, then all the series that we are considering are absolutely convergent. Using the fact that a product of absolutely convergent series converges to the Cauchy product of the series, we get the desired identity for the weight vector tx. From Lemma 3.13, it follows that the functions Φ are analytic functions of t on the set $\rho_{\mathcal{G}}(tx) = t\rho_{\mathcal{G}}(x) < 1$. Hence, by the assumption that $\rho_{\mathcal{G}}(x) < 1$ and by uniqueness of the analytic continuation, the desired identity follows also for the weight vector x. □

3.8 Proofs of the results for the Ising model

We will need the following classical inequality of Griffiths [28]. Note that this is the only external result on the Ising model that we use in this thesis.

Lemma 3.11 (Griffiths, [28]). *Let $\mathcal{G} = (V, E)$ be any graph and let $J = (J_e)_{e\in E}$, $J' = (J'_e)_{e\in E}$ be such that $|J_e| \leq J'_e$ for all $e \in E$. Then, for all $A \subset V$ and $\beta > 0$,*

$$\langle \sigma_A \rangle_{\mathcal{G},\beta} \leq \langle \sigma_A \rangle'_{\mathcal{G},\beta},$$

where the correlation functions are computed for the Ising model with free boundary conditions and coupling constants J and J' respectively.

Remark 3.12. *The positive boundary conditions arise from the free boundary conditions in a very natural limiting procedure, namely by taking the limit with the coupling constants between any two boundary vertices going to infinity and conditioning on one boundary spin to be positive. In particular, the lemma above implies that the correlation functions of the Ising model with positive boundary conditions dominate the correlation functions of the model with free boundary conditions. Moreover, if the coupling constants are positive, and $\mathcal{H}$ is a subgraph of $\mathcal{G}$, then the correlation functions of the Ising model on $\mathcal{G}$ with free boundary conditions dominate the corresponding correlation functions on $\mathcal{H}$.*

Recall the notation of Theorem 2.7 from Section 2.1.2.

Proof of Theorem 3.1. Let $\mathcal{G}$ be a rectangle and let $v_\bullet$ be an additional vertex in the unbounded face of $\mathcal{G}$. Fix a star $\mathcal{S} = \mathcal{S}(u, v)$, such that both its arms enter $\mathcal{G}$ through the same edge, and the weight vector x' defined as in (3.12) is equal to x'_γ restricted to $\mathcal{G}$ (See Figure 3.2). If $B \subset \partial\mathcal{G}$, then we will always choose $\mathcal{S}(B)$ in such a way that there are no edge crossings between $\mathcal{S}(B)$ and $\mathcal{G} \cup \mathcal{S}$, and moreover $\mathrm{sgn}(\mathcal{S}(B), \mathcal{S}) = 1$.

From Theorem 2.11, we know that the operator norm of $\Lambda_\mathcal{G}(x')$ is bounded above by $\delta = \frac{\tanh\beta}{\tanh\beta_c} < 1$, and hence also $\rho_\mathcal{G}(x') < 1$. Therefore, using Lemma 3.4 and Corollary 3.7, we get that for all $B \subset \partial\mathcal{G}$,

$$\begin{aligned} Z_{\mathcal{G},\mathcal{S}\cup\mathcal{S}(B)}(x') &= \Phi_{\mathcal{G},\mathcal{S}\cup\mathcal{S}(\mathcal{B})}(x') Z_\mathcal{G}(x') = \Big(\Phi_{\mathcal{G},\mathcal{S}}(x')\Phi_{\mathcal{G},\mathcal{S}(\mathcal{B})}(x') + \epsilon\Big) Z_\mathcal{G}(x') \\ &= \Phi_{\mathcal{G},\mathcal{S}}(x') Z_{\mathcal{G},\mathcal{S}(B)}(x') + \epsilon \cdot Z_\mathcal{G}(x'), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \epsilon &= \sum_{\substack{\pi\in\mathcal{P}(\{u,v\}\cup B)\\ \{u,v\}\notin\pi}} \mathrm{sgn}(\pi) \prod_{p\in\pi} \Phi_{\mathcal{G},\mathcal{S}(p)}(x') \\ &= \sum_{b,b'\in B} \sum_{\substack{\pi\in\mathcal{P}(\{u,v\}\cup B)\\ \{u,b\},\{v,b'\}\in\pi}} \mathrm{sgn}(\pi) \prod_{p\in\pi} \Phi_{\mathcal{G},\mathcal{S}(p)}(x') \\ &= \sum_{b,b'\in B} \pm\Phi_{\mathcal{G},\mathcal{S}(u,b)}(x')\Phi_{\mathcal{G},\mathcal{S}(v,b')}(x') \sum_{\pi\in\mathcal{P}(B\setminus\{b,b'\})} \mathrm{sgn}(\pi) \prod_{p\in\pi} \Phi_{\mathcal{G},\mathcal{S}(p)}(x') \\ &= \sum_{b,b'\in B} \pm\Phi_{\mathcal{G},\mathcal{S}(u,b)}(x')\Phi_{\mathcal{G},\mathcal{S}(v,b')}(x')\Phi_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x'). \end{aligned}$$

The second equality follows from Lemma 3.6 and the last one from Lemma 3.5. Hence,

$$
\begin{aligned}
|\epsilon| &\leq \sum_{b,b'\in B} |\Phi_{\mathcal{G},\mathcal{S}(u,b)}(x')| \cdot |\Phi_{\mathcal{G},\mathcal{S}(v,b')}(x')| \cdot |\Phi_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x')| \\
&\leq C_\beta \delta^{d(u,B)+d(v,B)} \sum_{b,b'\in B} \left|\Phi_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x')\right|
\end{aligned}
$$

where C_β is a constant depending only on β and not on $\mathcal{G}$, and d is the graph distance. In the second inequality, we used the fact that the functions Φ can be expressed in terms of entries of $\Lambda_{\mathcal{G}}(x')$ (as it was done in the proof of Theorem 2.7), and therefore the bound follows from the bound on the operator norm of $\Lambda_{\mathcal{G}}(x')$. Hence by Lemma 3.4,

$$
\begin{aligned}
|\epsilon \cdot Z_{\mathcal{G}}(x')| &\leq C_\beta \delta^{d(u,B)+d(v,B)} \sum_{b,b'\in B} \left|\Phi_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x') Z_{\mathcal{G}}(x')\right| \\
&= C_\beta \delta^{d(u,\partial\mathcal{G})+d(v,\partial\mathcal{G})} \sum_{b,b'\in B} \left|Z_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x')\right| \\
&\leq C_\beta \delta^{d(u,\partial\mathcal{G})+d(v,\partial\mathcal{G})} \sum_{b,b'\in B} Z_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x).
\end{aligned}
$$

To conclude the last inequality, we used the fact that $Z_{\mathcal{G},\mathcal{S}(B)}(x')$ counts the same weighted graphs as $Z_{\mathcal{G},\mathcal{S}(B)}(x)$, but some of them get negative signs, whereas in the latter generating function they all get positive signs. Plugging (3.16) into (3.12) and using (3.11), we obtain

$$
\langle \sigma_u \sigma_v \rangle^+_{\mathcal{G},\beta} = \Phi_{\mathcal{G},\mathcal{S}}(x') \langle \sigma_{u^*} \sigma_{v^*} \rangle^{\text{free}}_{\mathcal{G}^*,\beta^*} + \epsilon' = \Phi_{\mathcal{G},\gamma}(x'_\gamma) \langle \sigma_{u^*} \sigma_{v^*} \rangle^{\text{free}}_{\mathcal{G}^*,\beta^*} + \epsilon',
$$

where

$$
\begin{aligned}
|\epsilon'| &\leq C_\beta \delta^{d(u,\partial\mathcal{G})+d(v,\partial\mathcal{G})} \Big(\sum_{B\subset\partial\mathcal{G}} \sum_{b,b'\in B} Z_{\mathcal{G},\mathcal{S}(B\setminus\{b,b'\})}(x) \Big) \Big/ \Big(\sum_{B\subset\partial\mathcal{G}} Z_{\mathcal{G},\mathcal{S}(B)}(x) \Big) \\
&= C_\beta \delta^{d(u,\partial\mathcal{G})+d(v,\partial\mathcal{G})} \Big(\sum_{b,b'\in\partial\mathcal{G}} \sum_{B\subset\partial\mathcal{G}\setminus\{b,b'\}} Z_{\mathcal{G},\mathcal{S}(B)}(x) \Big) \Big/ \Big(\sum_{B\subset\partial\mathcal{G}} Z_{\mathcal{G},\mathcal{S}(B)}(x) \Big) \\
&\leq C_\beta \delta^{d(u,\partial\mathcal{G})+d(v,\partial\mathcal{G})} |\partial\mathcal{G}|^2.
\end{aligned}
$$

By Lemma 3.11 and Remark 3.12, $\langle \sigma_{u^*} \sigma_{v^*} \rangle^{\text{free}}_{\mathcal{G}^*,\beta^*}$ increases to $\langle \sigma_{u^*} \sigma_{v^*} \rangle^{\text{free}}_{\mathbb{Z}^{2*},\beta^*}$ when $\mathcal{G} \to \mathbb{Z}^2$. Note that by Remark 3.8,

$$
\Phi_{\mathcal{G},\mathcal{S}(u,v)}(x) = \Phi_{\mathcal{G},\gamma}(x) = \sum_{r=1}^{\infty} \sum_{\ell\in\mathcal{L}_r^{uu^*}(\mathcal{G}_\gamma)} w(\ell; x),
$$

where $\mathcal{L}_r^{uu^*}(\mathcal{G}_\gamma)$ is as in (2.48). Therefore, by Theorem 2.7, $\Phi_{\mathcal{G},\gamma}(x'_\gamma)$ converges to $\Phi_{\mathbb{Z}^2,\gamma}(x'_\gamma)$ as $\mathcal{G} \to Z^2$. Using the fact that $\epsilon' \to 0$ as $\mathcal{G} \to \mathbb{Z}^2$, we finish the proof. □

We can now prove Corollary 3.2, and hence provide a complete picture of the magnetic phase transition of the Ising model on the square lattice.

Proof of Corollary 3.2. By Theorems 2.7 and 3.1, and Remark 3.8,

$$\langle\sigma_u\sigma_v\rangle^{\text{free}}_{\mathbb{Z}^2,\beta} = \Phi_{\mathbb{Z}^2,\gamma}(x'_\gamma)\langle\sigma_{u^*}\sigma_{v^*}\rangle^{+}_{\mathbb{Z}^{2*},\beta^*} \text{ and } \langle\sigma_u\sigma_v\rangle^{+}_{\mathbb{Z}^2,\beta} = \Phi_{\mathbb{Z}^2,\gamma}(x'_\gamma)\langle\sigma_{u^*}\sigma_{v^*}\rangle^{\text{free}}_{\mathbb{Z}^{2*},\beta^*}$$

for $\beta \in (0, \beta_c)$. From the high temperature expansion, we know that for finite rectangles, any two-point function with free boundary conditions is strictly positive. Hence, by Lemma 3.11 and Remark 3.12, the infinite volume correlation function is also strictly positive. Using again Lemma 3.11 and the identities above, we obtain that for all $u, v \in \mathbb{Z}^2$ and $\beta \in (0, \beta_c)$,

$$1 \leq \frac{\langle\sigma_u\sigma_v\rangle^{+}_{\mathbb{Z}^2,\beta}}{\langle\sigma_u\sigma_v\rangle^{\text{free}}_{\mathbb{Z}^2,\beta}} = \frac{\langle\sigma_{u^*}\sigma_{v^*}\rangle^{\text{free}}_{\mathbb{Z}^{2*},\beta^*}}{\langle\sigma_{u^*}\sigma_{v^*}\rangle^{+}_{\mathbb{Z}^{2*},\beta^*}} \leq 1.$$

$\square$